Urban and Hinterland Evolution under Growing Population Pressure

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Abstract

An integrated mathematical model for the evolution of urban structure and population is presented.
The city configuration consists of an occupation number representation of different kinds of buildings such as lodgings and factories distributed over a grid of plots, and the population configuration describes the distribution of the population between city (c) and hinterland (h).
The dynamics of the total configuration is governed by motivation – dependent transition rates between neighbouring configurations. Equations of evolution on the stochastic level (mastereqatuion) and deterministic level (quasi-meanvalue equations) can thereupon be derived.
We focus on that sector of the model describing the population dynamics between hinterland (h) and city (c). Under the assumption of equal net birth rates in (c) and (h), and for given growth of the total population $P(t)$, the dynamics of the population shares between (h) and (c) can be treated explicitely in terms of a time dependent evolution potential.
One can distinguish between the two main cases of "constructive competition between (c) and (h)" and "worsending balance between (c) and (h)". In the first case a stabilisation of the population shares in c and h takes place, whereas in the second case a dramatic migratory phase transition sets in, namely a sudden rush of the population from the depleting hinterland to the overcrowding city.

KEYWORDS:

1. Integration of Urban and Population Dynamics
2. Motivation Dependent Transition Rates
3. Master Equation
4. Quasimeanvalue Equations
5. Migratory Phase-Transitions

I. General Design Principles

The design makes use of the general modelling procedure of "Sociodynamics" which is exhibited in detail in [1]. It consists of the following steps:

1. **Search for a "window of perception"**
   This search leads to the choice of a model-specific set of appropriate *order parameters* or *key variables*. It is anticipated that these variables satisfy an *approximately selfcontained dynamics* under general boundary conditions.
2. **The elementary steps of dynamics**
   The elementary steps lead to small changes of the orderparameters. They are provided by *motivation-driven probabilistic transition* rates composed of *mobility* and *utility terms*. These establish the link between the *microlevel* of orderparameters.
3. **The equations of evolution**
   As a consequence of the elementary dynamic steps *equations of motion for the orderparameters* can now be set up. They can be derived on the *stochastic level (master equation)* or on the *deterministic level (quasimeanvalue equations)* for the mean evolution of stochastic trajectories.
4. **Simulation of Characteristic Scenarios**
   The transition rates entering the dynamic equations contain certain *control-* and *trend*-parameters. After their *calibration* characteristic (realistic or virtual) scenarios can be simulated by solving the equations. These can be compared with empirical data.
5. **What can be learned from the model?**

   a) The urban evolution depends decisively on the *trendparameters* in the transition rates. They are measures of the influence of conditions of the *landscape*, of socio-economic *preferences*, of *neighbourhood relations* between building plots, and of *migratory trends* in the population.
   b) Model simulations of urban dynamics not only include the imitation of the *real evolution* but also *virtual evolutions* and *forecasts* resulting from the choice of different trend parameter sets.
   c) The urban evolution is *path-dependent*. Even if the trendparameters coincide, small deviations of the initial conditions may lead (at instable situations) to *diverging further evolution paths*, due to *inherent nonlinearities*. Simulations help to detect the situations where *bifurcations* occur.
   d) The *migration of population* and the development of *city and hinterland* are *interrelated processes*. In particular, *migratory phase-transitions*, e.g. a population rush from hinterland to city can occur and are analyzable in terms of the model.

II. The Integrated model for Urban and Population Evolution

1. The Key-Variables

a) **Material Variables: The City Configuration**
The city "c" and hinterland "h" are tessellated into a grid of *"plots"* or *"sites"* with coordinates $i(i_1, i_2), j(j_1, j_2) \ldots$. Each plot i has a *capacity* C_i, at a given time, to erect different kinds "k" of buildings – such as lodgings L, factories F, ... – on that plot.
The integer $m_i^{(k)}$ denotes the *number of construction units of kind* k erected on plot i.
The set of variables $m_i^{(k)}$ and capacities C_i:

$$\mathbf{m} = \{\ldots; m_i^{(F)}, \ldots m_i^{(L)}, \ldots C_i; \ldots; m_j^{(F)} \ldots, m_j^{(L)}, \ldots C_j; \ldots\} \tag{1}$$

is denoted as *city configuration.* It provides an *occupation number representation* of the state of development of the city. There hold the evident relations:

$$\sum_K m_i^{(k)} \leq C_i \tag{2}$$

$$C_c = \sum_{i \in c} C_i \tag{3}$$

$$C_h = \sum_{i \in h} C_i \tag{4}$$

where C_c and C_h are the *total capacities of city and hinterland*, respectively.

b) **Personal Variables: The Population Configuration**
Let $\mathcal{P}^{(\alpha)}$ denote *sub-populations* differentiated by profession, origin, background, etc.
Let $N_i^{(\alpha)}$ be the *number of members of subpopulation* $\mathcal{P}^{(\alpha)}$ living on plot i.
The set of variables

$$\mathbf{N} = \{\ldots; \ldots, N_i^{(\alpha)} \ldots N_i^{(\beta)}; \ldots N_j^{(\alpha)} \ldots N_j^{(\beta)}; \ldots\} \tag{5}$$

is denoted as *population configuration.* It describes the distribution of the sub-populations $\mathcal{P}^{(\alpha)}$ over the sites $i, \ldots, j, \ldots$, of city and hinterland.
Aggregated or global personal variables:

$$N_c = \sum_{\alpha, i \in c} N_i^{(\alpha)} = \text{ number of city inhabitants} \tag{6a}$$

$$N_h = \sum_{\alpha, i \in h} N_i^{(\alpha)} = \text{ number of hinterland inhabitants} \tag{6b}$$

$$P = pP_0 = N_c + N_h = \text{ number of total population with } P_0 = P(t=0) \tag{6c}$$

For later use we introduce the population shares n_c and n_n by

$$N_c = n_c P\,;\; N_h = n_h P\,;\; \text{with } \; n_c + n_h = 1 \tag{7a}$$

and

$$N = N_c - N_h = nP\,;\; \text{with } \; n = n_c - n_h \tag{7b}$$

where

$$-P \leq N \leq +P\,; \quad -1 \leq n \leq +1 \tag{7c}$$

2. Motivation-Driven Probabilistic Transition Rates

The *elementary steps* of city- and population-evolution are the following:

$$\mathbf{m} = \{\ldots, m_i^{(k)} \ldots\} \Rightarrow \mathbf{m}_{i\pm}^{(k)} = \{\ldots, (m_i^{(k)} \pm 1), \ldots\} \tag{8a}$$

$$\mathbf{N} = \{\ldots, N_i^{(\alpha)}, \ldots\} \Rightarrow \mathbf{N}_{i\pm}^{(\alpha)} = \{\ldots, (N_i^{(\alpha)} \pm 1), \ldots\} \tag{8b}$$

$$\mathbf{N} = \{\ldots, N_j^{(\alpha)}, \ldots, N_i^{(\alpha)} \ldots\} \Rightarrow \mathbf{N}_{ji}^{(\alpha)} = \{\ldots, (N_j^{(\alpha)} + 1), \ldots, (N_i^{(\alpha)} - 1), \ldots\} \tag{8c}$$

The meaning of the steps

(8a): one unit of kind k is erected or torn down on plot i

(8b): birth or death of one member of $\mathcal{P}^{(\alpha)}$ on site i

(8c): migration of one member of $\mathcal{P}^{(\alpha)}$ from site i to site j.

The *transition rates* engendering the steps (8) are:

$$\begin{aligned} w_i^k(\mathbf{m}_{i+}^{(k)}, \mathbf{m}; \mathbf{N}) &\equiv w_{i\uparrow}^k(\mathbf{m}; \mathbf{N}) \\ w_i^k(\mathbf{m}_{i-}^{(k)}, \mathbf{m}; \mathbf{N}) &\equiv w_{i\downarrow}^k(\mathbf{m}; \mathbf{N}) \end{aligned} \tag{9a}$$

$$\begin{aligned} w_{i+}^\alpha(\mathbf{m}; \mathbf{N}_{i+}^{(\alpha)}, \mathbf{N}) &\equiv w_{i\uparrow}^\alpha(\mathbf{m}, \mathbf{N}) \\ w_{i-}^\alpha(\mathbf{m}; \mathbf{N}_{i-}^{(\alpha)}, \mathbf{N}) &\equiv w_{i\downarrow}^\alpha(\mathbf{m}, \mathbf{N}) \end{aligned} \tag{9b}$$

$$\begin{aligned} w_{ji}^\alpha(\mathbf{m}; \mathbf{N}_{ji}^{(\alpha)}, \mathbf{N}) &\equiv w_{ji}^\alpha(\mathbf{m}, \mathbf{N}) = N_i^{(\alpha)} p_{ji}^{(\alpha)}[\mathbf{m}, \mathbf{N}] \\ &\text{where } p_{ji}^\alpha = \text{ individual migration rate from } i \text{ to } j. \end{aligned} \tag{9c}$$

In an explicit version of the model a concrete choice of the form of all transition rates (9a), (9b), (9c) must be made. Thereupon both sectors, the city and the population evolution, can be treated quantitatively. For details see [1] and [2]. In this lecture we only treat the *population sector* of the model (see III).

3. Evolution Equations

a) **The Stochastic Level: The Master Equation**

Definition

$$P(\mathbf{m};\mathbf{N};t) = \text{ probability that configuration } \{\mathbf{m},\mathbf{N}\} \text{ is realized at time } t \tag{10}$$

Normalization

$$\sum_{\mathbf{m},\mathbf{N}} P(\mathbf{m};\mathbf{N};t) = 1 \tag{11}$$

The Master Equation:

$$\begin{aligned}
\frac{dP(\mathbf{m};\mathbf{N};t)}{dt} &= \sum_k \sum_j \left[w^k_{j\uparrow}(\mathbf{m}^{(k)}_{j-};\mathbf{N})P(\mathbf{m}^{(k)}_{j-};\mathbf{N};t) - w^k_{j\uparrow}(\mathbf{m};\mathbf{N})P(\mathbf{m};\mathbf{N};t)\right] \\
&+ \sum_k \sum_j \left[w^k_{j\downarrow}(\mathbf{m}^{(k)}_{j+};\mathbf{N})P(\mathbf{m}^{(k)}_{j+};\mathbf{N};t) - w^k_{j\downarrow}(\mathbf{m};\mathbf{N})P(\mathbf{m};\mathbf{N};t)\right] \\
&+ \sum_\alpha \sum_{i,j} \left[w^\alpha_{ji}(\mathbf{m};\mathbf{N}^{(\alpha)}_{ij})P(\mathbf{m};\mathbf{N}^{(\alpha)}_{ij};t) - w^\alpha_{ji}(\mathbf{m};\mathbf{N})P(\mathbf{m};\mathbf{N};t)\right] \\
&+ \sum_\alpha \sum_{i} \left[w^\alpha_{i\uparrow}(\mathbf{m};\mathbf{N}^{(\alpha)}_{i-})P(\mathbf{m};\mathbf{N}^{(\alpha)}_{i-};t) - w^\alpha_{i\uparrow}(\mathbf{m};\mathbf{N})P(\mathbf{m};\mathbf{N};t)\right] \\
&+ \sum_\alpha \sum_{i} \left[w^\alpha_{i\downarrow}(\mathbf{m};\mathbf{N}^{(\alpha)}_{i+})P(\mathbf{m};\mathbf{N}^{(\alpha)}_{i+};t) - w^\alpha_{i\downarrow}(\mathbf{m};\mathbf{N})P(\mathbf{m};\mathbf{N};t)\right]
\end{aligned} \tag{12}$$

Terms on the r.h.s. of (12):

1. Line: probability flows by erection processes on all sites
2. Line: " " by tearing down processes on all sites
3. Line: " " by migration processes between all sites
4. Line: " " by birth processes on all sites
5. Line: " " by death processes on all sites.

b) **The Deterministic Level: Quasimeanvalue-Equations**

Quasimeanvalues $\hat{N}^{(\alpha)}_i(t); \hat{m}^{(k)}_j(t)$ describe the mean evolution of the stochastic variables $N^{(\alpha)}_i, m^{(k)}_j$ for a bundle of stochastic trajectories in configuration space starting from some initial condition $N^{(\alpha)}_i(t_0); m^{(k)}_j(t_0)$.

Quasimeanvalue-Equations are directly derivable utilizing the concept of *stochastic trajectories*. They read:

$$\frac{d\hat{m}_j^{(k)}}{dt} = w_{j\uparrow}^k(\hat{\mathbf{m}}, \hat{\mathbf{N}}) - w_{j\downarrow}^k(\hat{\mathbf{m}}, \hat{\mathbf{N}}) \tag{13}$$

$$\begin{aligned}\frac{d\hat{N}_j^{(\alpha)}}{dt} &= \sum_i w_{ji}^\alpha(\hat{\mathbf{m}}, \hat{\mathbf{N}}) - \sum_i w_{ij}^\alpha(\hat{\mathbf{m}}, \hat{\mathbf{N}}) \\ &+ w_{j\uparrow}^\alpha(\hat{\mathbf{m}}, \hat{\mathbf{N}}) - w_{j\downarrow}^\alpha(\hat{\mathbf{m}}, \hat{\mathbf{N}})\end{aligned} \tag{14}$$

Remark:
Quasimeanvalues $\hat{m}_j^{(k)}, \hat{N}_j^{(\alpha)}$ coincide approximately with the *meanvalues*

$$\bar{m}_j^{(k)}(t) = \sum_{\mathbf{m},\mathbf{N}} P(\mathbf{m}; \mathbf{N}; t) m_j^{(k)} \tag{15}$$

$$\bar{N}_j^{(\alpha)}(t) = \sum_{\mathbf{m},\mathbf{N}} P(\mathbf{m}; \mathbf{N}; t) N_j^{(\alpha)} \tag{16}$$

if the probability distribution $P(\mathbf{m}; \mathbf{N}; t)$ remains *unimodal* during its evolution with time.

III. A Simple Implementation of the Population-Sector: Global Treatment of City- and Hinterland-Population

1. The Global Population and Capacity Variables

$$\begin{aligned}\hat{N}_c(t) &= \text{number of city-inhabitants} \\ \hat{N}_h(t) &= \text{number of hinterland-inhabitants} \\ \hat{P}(t) = \hat{N}_c(t) + \hat{N}_h(t) &= \text{number of total population} \\ \hat{N}(t) = \hat{N}_c(t) - \hat{N}_h(t) &= \text{population difference}\end{aligned} \tag{17}$$

$$\begin{aligned}\hat{N}_c(t) &= \hat{n}_c(t)\hat{P}(t)\ ;\ \hat{N}_h(t) = \hat{n}_h(t)\hat{P}(t) \\ \hat{N}(t) &= \hat{n}(t)\hat{P}(t)\ ;\ \text{ with} \\ \hat{n}_c(t) + \hat{n}_h(t) &= 1\ ;\ \hat{n}_c(t) - \hat{n}_h(t) = \hat{n}(t). \\ \hat{P}(t) &= \hat{p}(t)\hat{P}_0,\ \text{ where } \hat{P}_0 \equiv \hat{P}(t = t_0) \\ -1 \le \hat{n}(t) &\le +1\ ;\ \hat{p}(t_0) = 1\end{aligned} \tag{18}$$

$\hat{n}_c(t)$ and $\hat{n}_h(t)$ are *population shares* and $\hat{n}(t)$ is the *share difference.*
Assumed form of *global capacities:*

$$C_c(t) = C_{c0} + \kappa_c \hat{N}_c(t) = P_0[c_{c0} + \kappa_c p(t)\hat{n}_c(t)]$$
$$C_h(t) = C_{h0} + \kappa_h \hat{N}_h(t) = P_0[c_{h0} + \kappa_h p(t)\hat{n}_h(t)] \tag{19}$$

C_{c0} and C_{h0} are extensive quantities:

$$C_{c0} = c_{c0}P_0; \quad C_{h0} = c_{h0}P_0, \tag{20}$$

where c_{co} and c_{h0} are size-independent parameters.

2. Global Personal Utilities and Transition Rates

Plausible assumption for the *"utility"* or *"attractiveness"* of city v_c and of hinterland v_h for potential migrants:

$$v_c = sP_0^{-1}C_c \quad v_h = sP_0^{-1}C_h \tag{21}$$

(If C_c, C_h, P_0 are extensive quantities, and s is independent of system size, then v_c and v_h are also size-independent.)
Transition Rates for migration between c and h, and for birth-death-processes within c and h.

$$w_{ch}(\hat{N}) = p_{ch}N_h = \text{migration rate from } h \text{ to } c \tag{22}$$
$$w_{hc}(\hat{N}) = p_{hc}N_c = \text{migration rate from } c \text{ to } h \tag{23}$$

with *individual transition rates*

$$p_{ch} = \nu \exp[v_c - v_h] \tag{24}$$
$$p_{hc} = \nu \exp[v_h - v_c] \tag{25}$$

and

$$w_{c\uparrow}(\hat{N}) = \beta_c \hat{N}_c = \text{birth rate in } c \tag{26}$$
$$w_{c\downarrow}(\hat{N}) = \delta_c \hat{N}_c = \text{ death rate in } c \tag{27}$$
$$w_{h\uparrow}(\hat{N}) = \beta_h \hat{N}_h = \text{birth rate in } h \tag{28}$$
$$w_{h\downarrow}(\hat{N}) = \delta_c \hat{N}_h = \text{ death rate in } h \tag{29}$$

3. Evolution Equations for the Population Configuration

The evolution equations for $\hat{N}_c$ and $\hat{N}_h$ are a special case of the quasimeanvalue equations (14). They read:

$$\begin{aligned}
\frac{d\hat{N}_c}{dt} &= w_{ch}(\hat{\mathbf{N}}) - w_{hc}(\hat{\mathbf{N}}) + w_{c\uparrow}(\hat{\mathbf{N}}) - w_{c\downarrow}(\hat{\mathbf{N}}) \\
&= p_{ch}(\hat{\mathbf{N}})\hat{N}_h - p_{hc}(\hat{\mathbf{N}})\hat{N}_c + \gamma_c \hat{N}_c \\
&= \nu \exp\left\{ s \left[(c_{c0} - c_{h0}) + P_0^{-1}(\kappa_c \hat{N}_c - \kappa_h \hat{N}_h) \right] \right\} \hat{N}_h \\
&- \nu \exp\left\{ -s \left[(c_{c0} - c_{h0}) + P_0^{-1}(\kappa_c \hat{N}_c - \kappa_h \hat{N}_h) \right] \right\} \hat{N}_c \\
&+ \gamma_c \hat{N}_c
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\frac{d\hat{N}_h}{dt} &= w_{hc}(\hat{\mathbf{N}}) - w_{ch}(\hat{\mathbf{N}}) + w_{h\uparrow}(\hat{\mathbf{N}}) - w_{h\downarrow}(\hat{\mathbf{N}}) \\
&= p_{hc}(\hat{\mathbf{N}})\hat{N}_c - p_{ch}(\hat{\mathbf{N}})\hat{N}_h + \gamma_h \hat{N}_h \\
&= -\nu \exp\left\{ s \left[(c_{c0} - c_{h0}) + P_0^{-1}(\kappa_c \hat{N}_c - \kappa_h \hat{N}_h) \right] \right\} \hat{N}_h \\
&+ \nu \exp\left\{ -s \left[(c_{c0} - c_{h0}) + P_0^{-1}(\kappa_c \hat{N}_c - \kappa_h \hat{N}_h) \right] \right\} \hat{N}_c \\
&+ \gamma_h \hat{N}_h
\end{aligned} \tag{31}$$

where

$$\gamma_c = (\beta_c - \delta_c) \quad ; \quad \gamma_h = (\beta_h - \delta_h) \tag{32}$$

are net birth rates of city and hinterland, respectively. The last lines of (30) and (31) follow by inserting (19), (20) and (21) into the individual transition rates (24) and (25).

4. The Case of Equal Net Birth Rates in City and Hinterland

We evaluate the case:

$$\gamma_c = \gamma_h = \gamma \tag{33}$$

where the net birth rate in city and hinterland

$$\gamma = \gamma(\hat{P}) \tag{34}$$

may depend on the total population $\hat{P}(t)$. In this case the evolution of the total population $\hat{P}(t)$ and of the population shares $\hat{n}_c(t), \hat{n}_h(t)$ can be *separated*!

4.1 Growth of Total Population $\hat{P}(t)$

Taking the sum of eqs. (30) and (31), one obtais with (33) *the separate equation* for the total population:

$$\frac{d\hat{P}(t)}{dt} = \gamma(\hat{P})\hat{P}(t), \quad \text{or} \quad \frac{d\hat{p}(t)}{dt} = \gamma(\hat{P})\hat{p}(t) \tag{35}$$

Eq. (35) is easily solved in the cases of

$$\gamma(\hat{P}) = \gamma_0 = \text{constant} \quad \text{(exponential growth)} \tag{36a}$$

or

$$\gamma(\hat{P}) = \gamma_0 \left(1 - \frac{\hat{P}}{P_s}\right) \quad \text{(logistic growth)} \tag{36b}$$

with the result:

$$\hat{P}(t) = \hat{p}(t)P_0 = \exp(\gamma_0 t)P_0 \tag{37a}$$

or

$$\hat{P}(t) = \hat{p}(t)P_0 = \frac{P_s \exp(\gamma_0 t)P_0}{\{P_s + P_0[\exp(\gamma_0 t) - 1]\}} \tag{37b}$$

4.2 Evolution of Population Shares $\hat{n}_c(t)$ and $\hat{n}_h(t)$

Inserting eq. (35) into eqs. (30) and (31), one now obtains equations for the population shares, i.e. for the *separated migratory process:*

$$\frac{d\hat{n}_c(t)}{dt} = -p_{hc}\hat{n}_c + p_{ch}\hat{n}_h \tag{38}$$

$$\frac{d\hat{n}_h(t)}{dt} = +p_{hc}\hat{n}_c - p_{ch}\hat{n}_h \tag{39}$$

where

$$p_{ch} = p_{ch}(\hat{n}, t) = \nu \exp[v_c - v_h] = \nu \exp[a(t) + b(t)\hat{n}] \tag{40}$$

$$p_{hc} = p_{hc}(\hat{n}, t) = \nu \exp[v_h - v_c] = \nu \exp[-a(t) - b(t)\hat{n}] \tag{41}$$

with coefficients

$$a(t) = s\left[(c_{c0} - c_{h0}) + \frac{1}{2}(\kappa_c - \kappa_h)p(t)\right] \tag{42}$$

and

$$b(t) = \frac{1}{2}s(\kappa_c + \kappa_h)p(t) \tag{43}$$

which are time-dependent via $p(t)$ and are related by

$$a(t) = s(c_{c0} - c_{h0}) + \frac{(\kappa_c - \kappa_h)}{(\kappa_c + \kappa_h)}b(t) \tag{44}$$

The Equation for $\hat{n}(t) = \hat{n}_c(t) - \hat{n}_h(t)$ and the Evolution Potential $V(\hat{n}, t)$

Because of $(\hat{n}_c(t) + \hat{n}_h(t)) = 1$, the equations (38) and (39) for the population shares $\hat{n}_c(t)$ and $\hat{n}_h(t)$ reduce to *one equation* for the share difference $\hat{n}(t) = (\hat{n}_c(t) - \hat{n}_h(t))$:

$$\frac{d\hat{n}(t)}{dt} = (p_{ch} - p_{hc}) - (p_{hc} + p_{ch})\hat{n}(t) \tag{45}$$

Let us now introduce the *evolution potential*

$$\begin{aligned} V(\hat{n}; t) &\equiv V[\hat{n}; a(t), b(t)] \\ &= \nu e^{(a(t)+b(t)\hat{n})} \left[\frac{b(t)\hat{n} - 1}{b^2(t)} - \frac{1}{b(t)}\right] \\ &- e^{-(a(t)+b(t)\hat{n}} \left[\frac{b(t)\hat{n} + 1}{b^2(t)} + \frac{1}{b(t)}\right] \end{aligned} \tag{46}$$

which is slowly time-dependent due to the growth factor $\hat{p}(t)$ on which $a(t)$ and $b(t)$ depend.

Making use of $V(\hat{n}; t)$, the evolution equation (45) can be re-written in the form:

$$\frac{d\hat{n}(t)}{dt} = -\frac{\partial V(\hat{n}; t)}{\partial \hat{n}} \tag{47}$$

a) The Case of constructive competition between city and hinterland

In this case, $c_{0c} = c_{0c}P_0$ and $c_{0h} = c_{0h}P_0$ are of the same order of magnitude, and $\kappa_c \gtrapprox \kappa_h > 0$ are both positive. This means, that both, city and hinterland, react positively to growth of their population by extending their capacity (see eq. (19)). It can be shown, that then the condition

$$b(t) \geq 1; \quad |a(t)| < a^*(b) \tag{48}$$

for *three quasistationary states*, where $\hat{\hat{n}}_-$ and $\hat{\hat{n}}_+$ are stable, remains realized *for all times* $t > 0$, even if $p(t)$ grows. If $\hat{\hat{n}}_-$ existed at $t = t_0$, $\hat{n} \approx \hat{\hat{n}}_-$ will persist also for $t > t_0$.

b) The Case of "worsening balance" between city and hinterland
In this case c_{0c} and c_{0h} still have the same order of magnitude, but $\kappa_c > 0$, and $\kappa_h < 0$ holds. This means, that $c_h(t)$ *decreases* with growing $N_h(t)$, for instance if the cattle of the hinterland population over-grazes the pasture ground so that $c_h(t)$ is shrinking. Then a time t^* exists, such that for $t > t^*$

$$b(t) \geq 1; \quad |a(t)| > a^*(b) \tag{49}$$

holds. Only one quasistationary state survives. A *migratory phase transition takes place for $t \geq t^*$.*

Illustration of the Migratory Phasetransition by the timedependent Evolution Potential

The timedependent form of the evolution potential gives the best illustration of how the *migratory phase transition* comes about. In case b) of worsening balance between city and hinterland the trendparameters are such that for $0 < t < t^*$ condition (48), but for $t > t^*$ condition (49) is fulfilled. Condition (48) means that $V(\hat{n}, t)$ has *two minima and one maximum inbetween*, whereas (49) means that $V(\hat{n}, t)$ has *only one minimum*. If the system started in the one minimum $(\hat{\hat{n}}_-)$ = quasi-stationary state, which *disappears* at time $t = t^*$, a *dramatic phase transition* must take place at times $t \geq t^*$ leading the system into the only surviving quasistationary state $\hat{\hat{n}}_+$ corresponding to the only one surviving minimum of the potential. The *dynamics of this pase transition* is of course described by the equation (47).

Literature

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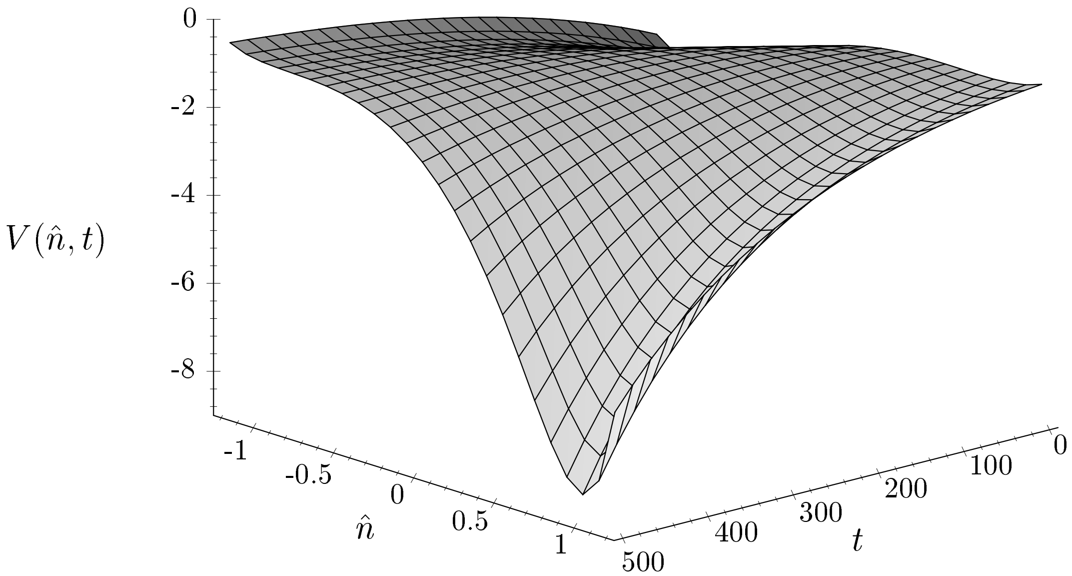

Figure 1: a) The evolution potential $V(\hat{n},t)$ in a case of worsening balance between c and h for logistic growth of the total population. Parameters: $s = 1,;$ $\nu = 1$; $(c_{c0} - c_{h0}) = -4.5$; $\kappa_c = 6$; $\kappa_h = -2$; $\gamma_0 = 0.001$; $P_0 = 1$; $P_S = 5. t^* = 458.93$; $\hat{\hat{n}}_-(0) = -0.985840$; $\hat{\hat{n}}_+(0) = 0.801759$.

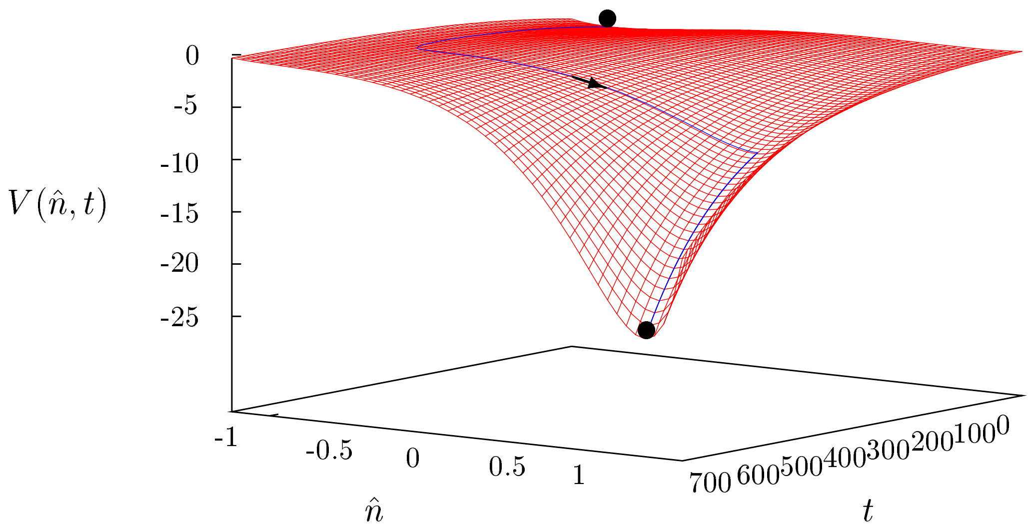

Figure 1: b) The evolution potential $V(\hat{n},t)$ with "system ball" representing the path of $\hat{n}(t)$ under conditions as in Fig. 1a.